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Adam M. Lowrance  
*Vassar College*

Kaitian Jin  
*Vassar College*

Eli Polston  
*Vassar College*

Yanjie Zheng  
*Vassar College*

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# ON THE TURAEV GENUS OF TORUS KNOTS

KAITIAN JIN, ADAM M. LOWRANCE, ELI POLSTON, AND YANJIE ZHENG

ABSTRACT. The Turaev genus and dealternating number of a link are two invariants that measure how far away a link is from alternating. We determine the Turaev genus of a torus knot with five or fewer strands either exactly or up to an error of at most one. We also determine the dealternating number of a torus knot with five or fewer strand up to an error of at most two. Additional bounds are given on the Turaev genus and dealternating number of torus links with five or fewer strands and on some infinite families of torus links on six strands.

## 1. INTRODUCTION

The Turaev surface of a link diagram was first constructed by Turaev [Tur87] to give an alternate method of proving Kauffman [Kau87], Murasugi [Mur87], and Thistlethwaite's [Thi88] theorem that the Jones polynomial gives a lower bound on the crossing number of a link. Specifically, if  $L$  is a link with diagram  $D$ , crossing number  $c(L)$ , Turaev surface of genus  $g_T(D)$ , and Jones polynomial  $V_L(t)$ , then Turaev proved that

$$\text{span } V_L(t) + g_T(D) \leq c(L).$$

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus [DFK<sup>+</sup>08] defined the Turaev genus  $g_T(L)$  of the link  $L$  to be the minimum genus of the Turaev surface of any link diagram of  $D$ . Turaev showed that the genus  $g_T(D)$  of the Turaev surface is zero if and only if  $D$  is a connected sum of alternating link diagrams, and consequently the Turaev genus of a link is zero if and only if the link is alternating. One can view Turaev genus as a filtration on links where links with large Turaev genus can be interpreted as being far away from alternating.

In this article, we compute the Turaev genus of several infinite families of torus knots on six or fewer strands. For positive, coprime integers  $p$  and  $q$ , let  $T_{p,q}$  be the  $(p, q)$ -torus knot. Since the Turaev genus of a link and its mirror are equal, it is enough to only consider positive torus knots. Because  $T_{2,q}$  is alternating for all  $q$ , it follows that  $g_T(T_{2,q}) = 0$ . Abe and Kishimoto [AK10] and Lowrance [Low11] proved that the Turaev genus of torus knots on three strands is given by

$$g_T(T_{3,3n+i}) = n$$

for each  $n \geq 0$  and for  $i = 0, 1$ , or  $2$ . The following theorem gives the Turaev genus of all torus knots on four or five strands up to an error of at most one and gives the Turaev genus of an infinite family of six stranded torus knots.

**Theorem 1.1.** *For each non-negative integer  $n$  and for  $j = 2, 3$ , and  $4$ ,*

$$\begin{aligned} g_T(T_{4,4n+1}) &= 2n, & g_T(T_{4,4n+3}) &= 2n + 1, \\ g_T(T_{5,5n+1}) &= 4n, & g_T(T_{6,6n+1}) &= 6n, \text{ and} \\ 4n + j - 2 &\leq g_T(T_{5,5n+j}) \leq 4n + j - 1. \end{aligned}$$

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Theorem 4.7 gives bounds on the Turaev genus of some infinite families of torus links on six or fewer strands.

Another way to measure how far a link  $L$  is from alternating is via its dealternating number  $\text{dalt}(L)$  [ABB<sup>+</sup>92]. The dealternating number  $\text{dalt}(D)$  of a link diagram  $D$  is the minimum number of crossing changes needed to transform  $D$  into an alternating diagram. The *dealternating number* of a link  $L$  is the minimum dealternating number of any diagram  $D$  of  $L$ . Abe and Kishimoto [AK10] show that  $g_T(L) \leq \text{dalt}(L)$  for any link  $L$ . The following theorem gives bounds on the dealternating numbers of some families of torus knots on six or fewer strands.

**Theorem 1.2.** *For each non-negative integer  $n$  and for  $j = 2, 3,$  and  $4,$*

$$\begin{aligned} 2n \leq \text{dalt}(T_{4,4n+1}) \leq 2n + 1, & & 2n + 1 \leq \text{dalt}(T_{4,4n+3}) \leq 2n + 2, \\ 4n \leq \text{dalt}(T_{5,5n+1}) \leq 4n + 1, & & 4n + j - 2 \leq \text{dalt}(T_{5,5n+j}) \leq 4n + j, \text{ and} \\ & & 6n \leq \text{dalt}(T_{6,6n+1}) \leq 6n + 2. \end{aligned}$$

We prove Theorems 1.1 and 1.2 in two steps: first we compute a lower bound coming from knot Floer homology, and second we find diagrams with the indicated Turaev genus or dealternating number. The knot Floer homology  $\widehat{HFK}(K)$  of a knot  $K$  is a categorification of the Alexander polynomial of  $K$  developed by Ozsváth and Szabó [OS04] and independently by Rasmussen [Ras03]. The knot Floer homology  $\widehat{HFK}(K)$  of  $K$  is a bigraded  $\mathbb{Z}$ -module with Alexander grading  $s$  and Maslov grading  $m$ . It decomposes into direct summands  $\widehat{HFK}(K) = \bigoplus_{s,m \in \mathbb{Z}} \widehat{HFK}_m(K, s)$ , and the symmetrized Alexander polynomial  $\Delta_K(t)$  of  $K$  can be recovered as a filtered Euler characteristic of knot Floer homology:

$$\Delta_K(t) = \sum_{s,m \in \mathbb{Z}} (-1)^m \text{rank } \widehat{HFK}_m(K, s) \cdot t^s.$$

Define

$$\begin{aligned} \delta_{\max}(K) &= \max\{s - m \mid \widehat{HFK}_m(K, s) \neq 0\} \text{ and} \\ \delta_{\min}(K) &= \min\{s - m \mid \widehat{HFK}_m(K, s) \neq 0\}. \end{aligned}$$

The *width* of the knot Floer homology is defined as

$$\text{width } \widehat{HFK}(K) = \delta_{\max}(K) - \delta_{\min}(K) + 1.$$

Lowrance [Low08] proved that the width of the knot Floer homology of a knot gives a lower bound on its Turaev genus. Ozsváth and Szabó [OS05] gave an algorithm to compute the knot Floer homology of a torus knot (or any knot with a lens or  $L$ -space surgery) from its Alexander polynomial. We use the Ozsváth and Szabó algorithm to compute the width of the knot Floer homology  $\widehat{HFK}(T_{p,q})$  for many torus knots  $T_{p,q}$ .

**Theorem 1.3.** *For each positive integer  $p$  and non-negative integer  $n,$*

$$\begin{aligned} \text{width } \widehat{HFK}(T_{p,pn+1}) &= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor + 1 \text{ and} \\ \text{width } \widehat{HFK}(T_{p,pn-1}) &= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor - \left\lfloor \frac{p-1}{2} \right\rfloor + 1. \end{aligned}$$

Moreover for each non-negative integer  $n,$

$$\text{width } \widehat{HFK}(T_{5,5n+2}) = 4n + 1 \text{ and } \text{width } \widehat{HFK}(T_{5,5n+3}) = 4n + 2.$$

We conjecture that there is a recursive formula to compute  $\widehat{HFK}(T_{p,q})$ . The base case of the recursion is  $\widehat{HFK}(T_{1,q}) = 1$  for all  $q$ . Also, since  $T_{p,q} = T_{q,p}$ , it follows that  $\widehat{HFK}(T_{p,q}) = \widehat{HFK}(T_{q,p})$ . These two rules together with Equation 1.1 would give a way to evaluate  $\widehat{HFK}(T_{p,q})$  for any pair of positive coprime integers  $(p, q)$ . Theorem 1.3 implies the conjecture is true for all pairs of positive coprime integers  $(p, q)$  where  $q \equiv \pm 1 \pmod p$  or  $p \leq 6$ . A computer computation shows that the conjecture is also true when  $p$  and  $q$  are less than 250.

**Conjecture 1.4.** *Let  $p$  and  $q$  be positive coprime integers with  $p < q$ . Then*

$$(1.1) \quad \widehat{HFK}(T_{p,q}) - \widehat{HFK}(T_{p,q-p}) = \left\lfloor \frac{(p-1)^2}{4} \right\rfloor.$$

Assuming Conjecture 1.4, the best-case scenario for the Turaev genus of a torus knot is that it satisfies an analogous recursive relation. In this line of thinking, we ask the following question.

**Question 1.5.** *Is it true that*

$$g_T(T_{p,q}) - g_T(T_{p,q-p}) = \left\lfloor \frac{(p-1)^2}{4} \right\rfloor$$

for all positive coprime integers  $p$  and  $q$  with  $p < q$ ?

This paper is organized as follows. In Section 2, we recall some basic facts about the Turaev genus and dealternating number of a link. In Section 3, we discuss lower bounds on Turaev genus and dealternating number, and we prove Theorem 1.3. Finally, in Section 4, we prove Theorems 1.1 and 1.2.

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## 2. TURAEV GENUS AND DEALTERNATING NUMBER

In this section, we review some properties of the Turaev surface, the Turaev genus of a link, and the dealternating numbers of a link. Champanerkar and Kofman [CK14] wrote an excellent recent survey of Turaev genus.

Each crossing in a link diagram  $D$  has an  $A$ -resolution and a  $B$ -resolution, as in Figure 1. The collection of curves in the plane obtained by choosing either an  $A$ -resolution or a  $B$ -resolution for each crossing in  $D$  is called a *Kauffman state* of  $D$ . The *all- $A$  state* is the Kauffman state obtained by choosing an  $A$ -resolution for each crossing. Similarly, the *all- $B$  state* is the Kauffman state obtained by choosing a  $B$ -resolution for each crossing.

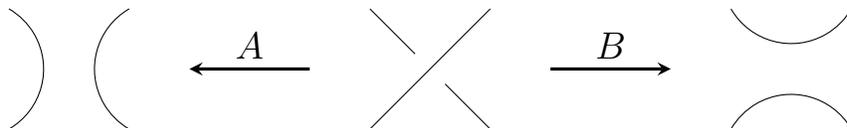


Figure 1: The  $A$ -resolution and  $B$ -resolution of a crossing.

Consider the link diagram  $D$  as embedded on the projection sphere  $S^2$ . Embed the all- $B$  state just inside the sphere and the all- $A$  state just outside the sphere. Away from neighborhoods of

crossings, connect the all- $B$  and all- $A$  state with bands. In a neighborhood of a crossing, connect the all- $B$  state with the all- $A$  state with a saddle, as in Figure 2. The resulting surface is a cobordism between the all- $B$  and all- $A$  states of  $D$ . The *Turaev surface*  $F(D)$  of  $D$  is obtained by capping off each of the boundary components with disks.

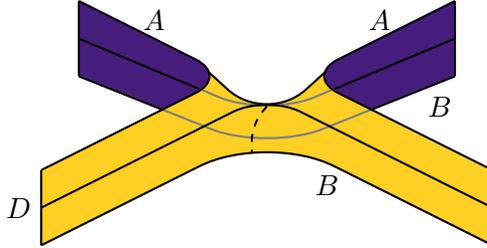


Figure 2: In a neighborhood of each crossing of  $D$ , the all- $B$  state is connected to the all- $A$  state with a saddle.

The Turaev surface of  $D$  is a Heegaard surface in  $S^3$ . The diagram  $D$  is alternating on the surface of  $F(D)$  (when viewed from one side of the surface). Let  $c(D)$  be the number of crossings in  $D$ , and let  $s_A(D)$  and  $s_B(D)$  be the number of components in the all- $A$  and all- $B$  states of  $D$  respectively. A link diagram can be thought of as a graph embedded in the plane where the crossings of  $D$  are the vertices of the graph and the arcs of  $D$  between the crossings are the edges of the graph. We say that  $D$  is connected if it is connected when thought of as a graph. We denote the genus of the Turaev surface of  $D$  by  $g_T(D)$ . If  $D$  is connected, then

$$(2.1) \quad g_T(D) = \frac{1}{2} (2 + c(D) - s_A(D) - s_B(D)).$$

Figure 3 depicts the Turaev surface of the  $(4, 5)$ -torus knot.

The Turaev surface has genus zero if and only if  $D$  is a connected sum of alternating diagrams. Armond and Lowrance [AL15] and independently Kim [Kim15] characterized link diagrams whose Turaev surface is genus one or two. Dasbach et. al. [DFK<sup>+</sup>08] proved that the Jones polynomial is an evaluation of the Bollobás-Riordan polynomial of a graph cellularly embedded in the Turaev surface. Dasbach and Lowrance [DL14] used the same graphs embedded in the Turaev surface to give a model for Khovanov homology.

The *Turaev genus*  $g_T(L)$  of a link  $L$  is defined as

$$g_T(L) = \min\{g_T(D) \mid D \text{ is a diagram of } L\}.$$

The Turaev genus of a link  $L$  is zero if and only if  $L$  is alternating. To compute the Turaev genus of a non-alternating link, one typically finds a diagram that is believed to be Turaev genus minimizing and then uses a computable obstruction or lower bound to prove that the diagram is in fact of minimal Turaev genus. Obstructions and lower bounds come from the Jones polynomial [DL16], Khovanov homology [CKS07], knot Floer homology [Low08], or from comparing certain concordance invariants [DL11]. The lower bound coming from knot Floer homology will be particularly important for us, and we will discuss it further in Section 3.

It has proven somewhat difficult to compute the Turaev genus of many infinite families of knots or links. Since a non-alternating link  $L$  with a genus one Turaev surface has  $g_T(L) = 1$ , it follows that non-alternating pretzel and Montesinos links are Turaev genus one. Abe [Abe09b] proved that any adequate link diagram is Turaev genus minimizing. Abe and Kishimoto [AK10] computed the

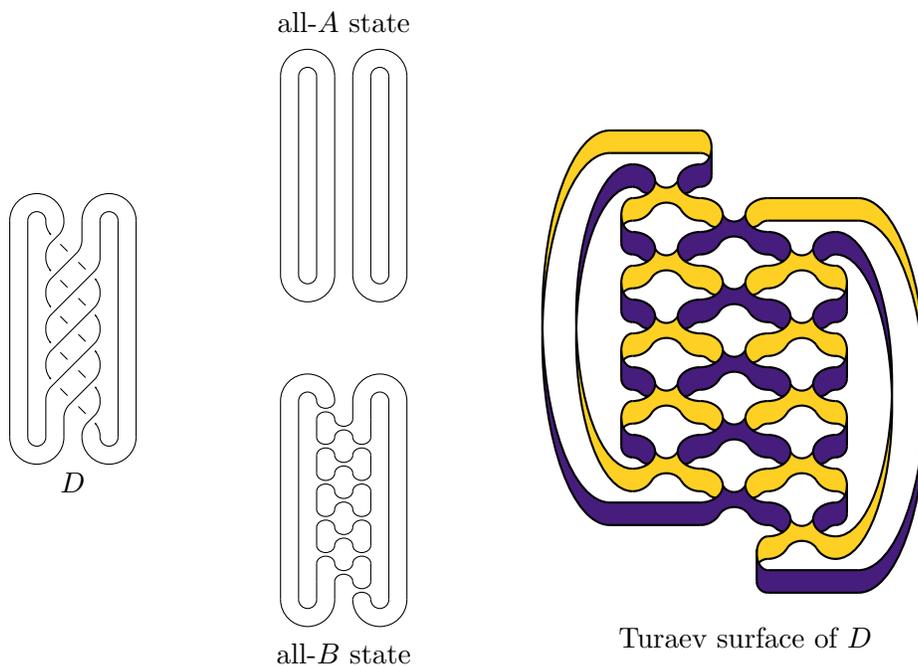


Figure 3: A diagram  $D$  of the  $(4, 5)$  torus knot, its all- $A$  and all- $B$  states, and its Turaev surface before capping off with disks.

Turaev genus of three stranded torus knots, and Lowrance [Low11] computed the Turaev genus of many closed 3-braids.

The *dealternating number*  $\text{dalt}(D)$  of a link diagram  $D$  is the minimum number of crossing changes needed to transform  $D$  into an alternating diagram. The *dealternating number*  $\text{dalt}(L)$  of the link  $L$  is defined by

$$\text{dalt}(L) = \min\{\text{dalt}(D) \mid D \text{ is a diagram of } L\}.$$

A link whose dealternating number is one is called *almost alternating*. Adams et. al. [ABB<sup>+</sup>92] defined the dealternating number of a link and studied almost alternating links. In particular, they proved that a prime almost alternating knot is either torus or hyperbolic. Abe [Abe09a] proved that the only almost alternating torus knots are  $T_{3,4}$ ,  $T_{3,5}$ , and their mirrors.

The dealternating number of a link has many of the same obstructions and lower bounds as Turaev genus, coming from the Jones polynomial [DL16], Khovanov homology [CK09], knot Floer homology [OS03], and from comparing concordance invariants [Abe09a]. Abe and Kishimoto [AK10] show that  $g_T(L) \leq \text{dalt}(L)$  for any link  $L$ . It is unknown whether there exists a link  $L$  such that  $g_T(L) < \text{dalt}(L)$ . Non-alternating pretzel links and Montesinos links [KL07, AK10] have dealternating number one. Moreover, the Turaev genus and dealternating number of three-stranded torus knots (and many closed 3-braids) agree [AK10].

The *alternation number*  $\text{alt}(D)$  of a link diagram  $D$  is the fewest number of crossing changes necessary to transform  $D$  into a (possibly non-alternating) diagram of an alternating link. The *alternation number*  $\text{alt}(L)$  of a link  $L$  is defined by

$$\text{alt}(L) = \min\{\text{alt}(D) \mid D \text{ is a diagram of } L\}.$$

Equivalently, one can think of the alternation number of a link as the Gordian distance between the link and the set of alternating links. The alternation number of a link was defined by Kawauchi [Kaw10]. An immediate consequence of the definition is that  $\text{alt}(L) \leq \text{dalt}(L)$  for any link  $L$ . Feller, Pohlmann, and Zentner [FPZ15] computed the alternation number of torus knots on four or fewer strands, and Baader, Feller, Lewark, and Zentner [BFLZ16] gave bounds on the alternation and dealternating number of some families of torus links on six or fewer strands. Our arguments in Section 4 resemble those in [FPZ15]. See [Low15] for more comparisons between Turaev genus, dealternating number, alternation number, and other related invariants.

### 3. KNOT FLOER WIDTH OF $T_{p,q}$

In this section, we compute the knot Floer width for the torus knot  $T_{p,q}$  where  $p = 5$  and  $q$  is arbitrary or where  $p$  is arbitrary and  $q = pn \pm 1$  for some positive integer  $n$ . The knot Floer width computation gives a lower bound on both Turaev genus and dealternating number.

Recall that for any knot  $K$ ,

$$\begin{aligned}\delta_{\max}(K) &= \max\{s - m \mid \widehat{HFK}_m(K, s) \neq 0\} \text{ and} \\ \delta_{\min}(K) &= \min\{s - m \mid \widehat{HFK}_m(K, s) \neq 0\}.\end{aligned}$$

Lowrance [Low08] proved the following theorem.

**Theorem 3.1.** *Let  $K$  be a knot of Turaev genus  $g_T(K)$ . Then*

$$\delta_{\max}(K) - \delta_{\min}(K) \leq g_T(K).$$

Let  $K$  be a knot such that there is an integral surgery on  $K$  that yields a lens space. Ozsváth and Szabó [OS05] prove that the nonzero coefficients of the Alexander polynomial are all  $\pm 1$  and that the knot Floer homology of  $K$  can be determined from the Alexander polynomial of  $K$ , as follows.

**Theorem 3.2.** *Let  $K$  be a knot in  $S^3$  such that there is an integral surgery on  $K$  yielding a lens space. Then there exists a sequence of integers  $s_{-k} < \cdots < s_k$  satisfying  $s_\ell = -s_{-\ell}$  such that the Alexander polynomial of  $K$  can be expressed as*

$$\Delta_K(t) = (-1)^k + \sum_{\ell=1}^k (-1)^{k-\ell} (t^{s_\ell} + t^{s_{-\ell}}).$$

For  $-k \leq \ell \leq k$  define

$$m_\ell = \begin{cases} 0 & \text{if } \ell = k, \\ m_{\ell+1} - 2(s_{\ell+1} - s_\ell) + 1 & \text{if } k - \ell \text{ is odd,} \\ m_{\ell+1} - 1 & \text{if } k - \ell > 0 \text{ is even.} \end{cases}$$

Then  $\widehat{HFK}_{m_\ell}(K, s_\ell) \cong \mathbb{Z}$  for each  $\ell$ , and  $\widehat{HFK}_m(K, s) = 0$  otherwise.

In order to compute the lower bound for  $g_T(K)$ , we only need the quantities  $\delta_\ell = s_\ell - m_\ell$  rather than the pairs  $(m_\ell, s_\ell)$ . Moreover, since  $\widehat{HFK}_m(K, s) \cong \widehat{HFK}_{m-2s}(K, -s)$ , it follows that we only need to consider  $\delta_\ell$  for  $\ell = 0, 1, \dots, k$ . These observations lead to the following corollary.

**Corollary 3.3.** *Let  $K$  be a knot in  $S^3$  such that there is an integral surgery on  $K$  yielding a lens space. Suppose that the Alexander polynomial of  $K$  is given by*

$$\Delta_K(t) = (-1)^k + \sum_{\ell=1}^k (-1)^{k-\ell} (t^{s_\ell} + t^{s-\ell}).$$

For  $\ell = 0, 1, \dots, k$ , define

$$\delta_\ell = \begin{cases} s_k & \text{if } \ell = k, \\ \delta_{\ell+1} + (s_{\ell+1} - s_\ell) - 1 & \text{if } k - \ell \text{ is odd,} \\ \delta_{\ell+1} - (s_{\ell+1} - s_\ell) + 1 & \text{if } k - \ell > 0 \text{ is even.} \end{cases}$$

Then

$$\delta_{\max}(K) = \max\{\delta_\ell \mid 0 \leq \ell \leq k\} \text{ and } \delta_{\min}(K) = \min\{\delta_\ell \mid 0 \leq \ell \leq k\}.$$

Since  $pq \pm 1$  surgery on the torus knot  $T_{p,q}$  is a lens space [Mos71], Corollary 3.3 and Theorem 3.1 can be used to give a lower bound on the Turaev genus of  $T_{p,q}$ . The symmetrized Alexander polynomial of the  $(p, q)$  torus knot is

$$(3.1) \quad \Delta_{T_{p,q}}(t) = t^{-(p-1)(q-1)/2} \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

**Example 3.4.** The Alexander polynomial of  $T_{4,5}$  is

$$\Delta_{T_{4,5}}(t) = t^{-6} - t^{-5} + t^{-2} - 1 + t^2 - t^5 + t^6,$$

and its knot Floer homology is given in Table 1. We have  $\delta_{\max}(T_{4,5}) = 6$  and  $\delta_{\min}(T_{4,5}) = 4$ , and thus the Turaev genus of  $T_{4,5}$  must be at least two.

$\widehat{HFK}(T_{4,5})$

$s \setminus m$	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
6													$\mathbb{Z}$
5												$\mathbb{Z}$	
4													
3													
2												$\mathbb{Z}$	
1													
0								$\mathbb{Z}$					
-1													
-2								$\mathbb{Z}$					
-3													
-4													
-5			$\mathbb{Z}$										
-6	$\mathbb{Z}$												

TABLE 1. The knot Floer homology of  $T_{4,5}$ .

In order to apply Corollary 3.3 to a knot  $K$ , one must express the Alexander polynomial of  $K$  as a Laurent polynomial. However, Equation 3.1 expresses the Alexander polynomial of  $T_{p,q}$  as a rational function. The following proposition gives Laurent polynomial formulas for  $\Delta_{T_{p,q}}(t)$  for certain values of  $p$  and  $q$ .

**Proposition 3.5.** *If  $p = 2k + 1$  is odd, then*

$$\Delta_{T_{p,pn+1}}(t) = 1 + \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^k \sum_{j=0}^{n-1} t^{\epsilon(p[(k-i+1)n-j]-i)} (t^{\epsilon i} - 1), \text{ and}$$

$$\Delta_{T_{p,pn-1}}(t) = 1 + \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^k \sum_{j=0}^{n-1} t^{\epsilon p[(k-i+1)n-j-1]} (t^{\epsilon i} - 1).$$

*If  $p = 2k$  is even, then*

$$\begin{aligned} \Delta_{T_{p,pn+1}}(t) &= (-1)^{n+1} \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^{k-1} \sum_{j=0}^{n-1} t^{\epsilon(p[(k-i+1)n-j]-kn-i)} (t^{\epsilon i} - 1) \\ &\quad + \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^{n-1} (-1)^{n-i-1} t^{\epsilon ik}, \text{ and} \end{aligned}$$

$$\begin{aligned} \Delta_{T_{p,pn-1}}(t) &= (-1)^{n+1} \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^{k-1} \sum_{j=0}^{n-1} t^{\epsilon(p[(k-i)n-j-1]+kn)} (t^{\epsilon i} - 1) \\ &\quad + \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^{n-1} (-1)^{n-i-1} t^{\epsilon ik}. \end{aligned}$$

*For the remaining torus knots on five strands, we have*

$$\begin{aligned} \Delta_{T_{5,5n+2}}(t) &= 1 + \sum_{\epsilon \in \{\pm 1\}} \sum_{j=0}^n t^{\epsilon(10n-5j+1)} (t^{\epsilon} - 1) \\ &\quad + \sum_{\epsilon \in \{\pm 1\}} \sum_{j=0}^{n-1} t^{\epsilon(5n-5j-4)} (t^{4\epsilon} - t^{3\epsilon} + t^{\epsilon} - 1), \text{ and} \end{aligned}$$

$$\begin{aligned} \Delta_{T_{5,5n+3}}(t) &= 1 + \sum_{\epsilon \in \{\pm 1\}} \sum_{j=0}^{n-1} t^{\epsilon(10n-5j+3)} (t^{\epsilon} - 1) \\ &\quad + \sum_{\epsilon \in \{\pm 1\}} \sum_{j=0}^n t^{\epsilon(5n-5j)} (t^{4\epsilon} - t^{3\epsilon} + t^{\epsilon} - 1). \end{aligned}$$

*Proof.* We only prove the result for  $T_{p,pn+1}$  where  $p = 2k + 1$  is odd. The other results follow from an analogous strategy. Since the formula for the symmetrized Alexander polynomial of  $T_{p,q}$  is

$$\Delta_{T_{p,q}}(t) = t^{-(p-1)(q-1)/2} \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)},$$

our approach is to show that  $t^{-(p-1)(q-1)/2}(t^{pq} - 1)(t - 1)$  is the product of  $(t^p - 1)(t^q - 1)$  and our formula.

Let  $p = 2k + 1$  and  $q = pn + 1$ . Then

$$(3.2) \quad (t^q - 1)(t^p - 1) \sum_{i=1}^k \sum_{j=0}^{n-1} t^{p[(k-i+1)n-j]-i} (t^i - 1)$$

$$(3.3) \quad = t^{pkn+pn} \sum_{i=1}^k (t^q - 1) t^{-ipn-i} \sum_{j=0}^{n-1} (t^p - 1) (t^{i-jp} - t^{-jp})$$

$$(3.4) \quad = t^{pkn+pn} \sum_{i=1}^k (t^q - 1) t^{-ipn-i} \sum_{j=0}^{n-1} (t^{i+p-jp} - t^{p-jp} - t^{i-jp} + t^{-jp})$$

$$(3.5) \quad = t^{pkn+pn} \sum_{i=1}^k (t^q - 1) t^{-ipn-i} (t^q - 1) (t^{p+i} - t^p - t^{p-n+i} + t^{p-n})$$

$$(3.6) \quad = t^{pkn+pn} \sum_{i=1}^k (t^{p+q-ipn} - t^{p+q} - t^{p+1+i} + t^{p+1} - t^{p+i} + t^{p-pn-ipn} - t^{p-pn-iq})$$

$$(3.7) \quad = t^{pkn+p+q} - t^{pkn+p+q-1} - t^{p+q} + t^{pn+p-k} + t^p - t^{p-k}.$$

In the above equation, (3.5) and (3.7) follow from their respective previous steps because the sums are telescoping. One can similarly prove that

$$\begin{aligned} & (t^p - 1)(t^q - 1) \sum_{i=1}^k \sum_{j=0}^{n-1} t^{-p[(k-i+1)n-j]+i} (t^{-i} - 1) \\ &= -t^{pn+p-k} + t^q + t^{p-k} - 1 - t^{1-pkn} + t^{-pkn}. \end{aligned}$$

Therefore

$$\begin{aligned} & (t^p - 1)(t^q - 1) \left( 1 + \sum_{\epsilon \in \{\pm 1\}} \sum_{i=1}^k \sum_{j=0}^{n-1} t^{\epsilon(p[(k-i+1)n-j]-i)} (t^{\epsilon i} - 1) \right) \\ &= (t^p - 1)(t^q - 1) + (t^{pkn+p+q} - t^{pkn+p+q-1} - t^{p+q} + t^{pn+p-k} + t^p - t^{p-k}) \\ & \quad + (-t^{pn+p-k} + t^q + t^{p-k} - 1 - t^{1-pkn} + t^{-pkn}) \\ &= t^{pkn+p+q} - t^{pkn+p+q-1} - t^{1-kpn} + t^{-kpn} \\ &= t^{-(p-1)(q-1)/2} (t^{pq} - 1)(t - 1), \end{aligned}$$

proving our result.  $\square$

We prove Theorem 1.3 using Corollary 3.3 and Proposition 3.5.

*Proof of Theorem 1.3.* The proof of this theorem breaks up into cases, one for each of the formulas appearing in Proposition 3.5. In each case, the symmetry of  $\widehat{HFK}(K)$  ensures that it is enough to consider the constant term and the  $\epsilon = 1$  terms in the sum. The strategy for each case is to apply Corollary 3.3 to Proposition 3.5. For notational convenience, we will reindex the  $\delta_\ell$ -terms in Corollary 3.3 to more closely match the indices of the sums in Proposition 3.5.

Suppose  $p = 2k + 1$ . The terms with positive degree in  $\Delta_{T_{p,pn+1}}(t)$  come in pairs. We reindex the terms in Corollary 3.3 as  $\delta_{i,j,1}$  and  $\delta_{i,j,2}$  to correspond to the  $t^i$  and 1 terms respectively in the sum with indices  $i$  and  $j$ . The initial  $\delta$ -term of Corollary 3.3 is  $\delta_{1,0,1} = pkn$ . Each  $\delta_{i,j,2}$  term occurs an odd number of times after the first step in the iteration. Hence  $\delta_{i,j,2} = \delta_{i,j,1} + i - 1$ . Each  $\delta_{i,j,1}$  term occurs an even number of times after the first step. For each  $i$ , we have  $\delta_{i,j+1,1} = \delta_{i,j,1} - p + 2i$  if  $0 < j < n - 1$  and  $\delta_{i+1,0,1} = \delta_{i,n-1,1} - p + 2i$ . Finally, if  $\delta_0$  is the term corresponding to the constant term in the Alexander polynomial, then  $\delta_0 = \delta_{k,n-1,1} - 1$ . Thus  $\delta_{\max}(T_{p,pn+1}) = \delta_{1,0,1}$ ,

$\delta_{\min}(T_{p,pn+1}) = \delta_0$  and

$$\begin{aligned} \delta_{\max}(T_{p,pn+1}) - \delta_{\min}(T_{p,pn+1}) &= n \sum_{i=1}^k p - 2i \\ &= n \sum_{i=1}^k 2i - 1 \\ &= nk^2 \\ &= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor. \end{aligned}$$

The terms with positive degree in  $\Delta_{T_{p,pn-1}}$  come in pairs. We reindex the terms in Corollary 3.3 as  $\delta_{i,j,1}$  and  $\delta_{i,j,2}$  to correspond to the  $t^i$  and 1 terms respectively in the sum with indices  $i$  and  $j$ . The initial  $\delta$ -term is  $\delta_{1,0,1} = pkn - 2k$ . Each  $\delta_{i,j,2}$  term occurs an odd number of times after the first step in the iteration. Hence  $\delta_{i,j,2} = \delta_{i,j,1} + i - 1$ . Each  $\delta_{i,j,1}$  term occurs an even number of steps after the initial term. For each  $i$ , we have  $\delta_{i,j+1,1} = \delta_{i,j,1} - p + 2i$  if  $0 < j < n - 1$  and  $\delta_{i+1,0,1} = \delta_{i,n-1,1} - p + 2i + 1$ . The term corresponding to the constant term in the Alexander polynomial is  $\delta_{k,n-1,2}$ . Thus  $\delta_{\max}(T_{p,pn-1}) = \delta_{1,0,1}$ ,  $\delta_{\min}(T_{p,pn-1}) = \delta_{k,n-1,1}$  and

$$\begin{aligned} \delta_{\max}(T_{p,pn-1}) - \delta_{\min}(T_{p,pn-1}) &= (n-1) \sum_{i=1}^k (p-2i) + \sum_{i=1}^k (p-2i-1) \\ &= (n-1) \sum_{i=1}^k (2i-1) + 2 \sum_{i=1}^{k-1} i \\ &= (n-1)k^2 + (k-1)k \\ &= nk^2 - k \\ &= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor - \left\lfloor \frac{p-1}{2} \right\rfloor. \end{aligned}$$

Let  $p = 2k$ . There are two sums in  $\Delta_{T_{p,pn+1}}(t)$ . In the first sum, the terms come in pairs. The terms will be labeled by  $\delta_{i,j,1}$  and  $\delta_{i,j,2}$  to correspond to the  $t^i$  and 1 terms respectively. In the second sum, we label each term as  $\tilde{\delta}_i$  to correspond with the term  $t^{ik}$ . The constant term is again labeled  $\delta_0$ . The initial  $\delta$ -term is  $\delta_{1,0,1} = (p-1)kn$ . Each  $\delta_{i,j,2}$  term occurs an odd number of steps after the initial term. Hence  $\delta_{i,j,2} = \delta_{i,j,1} + i - 1$ . Each  $\delta_{i,j,1}$  term occurs an even number of steps after the initial term. For each  $i$ , we have  $\delta_{i,j+1,1} = \delta_{i,j,1} - p + 2i$  if  $0 < j < n - 1$  and  $\delta_{i+1,0,1} = \delta_{i,n-1,1} - p + 2i$ . The term in the second sum with highest degree is the term indexed by  $n-1$ . We have  $\tilde{\delta}_{n-1} = \delta_{k-1,n-1,2} - k = \delta_{k-1,n-1,1} + (k-1) - 1 - k = \delta_{k-1,n-1,1} - 2$ . After that point we have  $\tilde{\delta}_{n-1-i} = \tilde{\delta}_{n-1}$  if  $i$  is even and  $\tilde{\delta}_{n-1-i} = \tilde{\delta}_{n-1} + k - 1$  if  $i$  is odd. Finally  $\delta_0 = \tilde{\delta}_{n-1} + k - 1$ . Thus  $\delta_{\max}(T_{p,pn+1}) = \delta_{1,0,1}$ ,  $\delta_{\min}(T_{p,pn+1}) = \tilde{\delta}_{n-1}$ , and

$$\begin{aligned} \delta_{\max}(T_{p,pn+1}) - \delta_{\min}(T_{p,pn+1}) &= n \sum_{i=1}^{k-1} p - 2i \\ &= 2n \sum_{i=1}^{k-1} i \end{aligned}$$

$$\begin{aligned}
&= nk(k+1) \\
&= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor.
\end{aligned}$$

There are two sums in  $\Delta_{T_{p,pn-1}}(t)$ . In the first sum, the terms come in pairs, and will be labeled by  $\delta_{i,j,1}$  and  $\delta_{i,j,2}$  to correspond to the  $t^i$  and 1 terms respectively. In the second sum, we label the term corresponding to  $t^{ik}$  by  $\tilde{\delta}_i$ . The term corresponding to the constant term in the Alexander polynomial is again labeled  $\delta_0$ . The initial  $\delta$ -term is  $\delta_{1,0,1} = (p-1)(kn-1)$ . Each  $\delta_{i,j,2}$  term occurs an odd number of steps after the initial term. Hence  $\delta_{i,j,2} = \delta_{i,j+1,1} - i + 1$ . Each  $\delta_{i,j,1}$  term occurs an even number of steps after the initial term. For each  $i$ , we have  $\delta_{i,j+1,1} = \delta_{i,j,1} - p + 2i$  if  $0 < j < n-1$  and  $\delta_{i+1,0,1} = \delta_{i,n-1,1} - p + 2i + 1$ . The term in the second sum with greatest degree is  $\tilde{\delta}_{n-1}$ . We have  $\tilde{\delta}_{n-1} = \delta_{k-1,n-1,2} - k + 1 = (\delta_{k-1,n-1,1} + k - 2) - k + 1 = \delta_{k-1,n-1,1} - 1$ . After that point we have  $\tilde{\delta}_{n-1-i} = \tilde{\delta}_{n-1}$  if  $i$  is even and  $\tilde{\delta}_{n-1-i} = \tilde{\delta}_{n-1} + k - 1$  if  $i$  is odd. Finally  $\delta_0 = \tilde{\delta}_{n-1} + k - 1$ . Thus  $\delta_{\max}(T_{p,pn-1}) = \delta_{1,0,1}$ ,  $\delta_{\min}(T_{p,pn-1}) = \tilde{\delta}_{n-1}$ , and

$$\begin{aligned}
\delta_{\max}(T_{p,pn-1}) - \delta_{\min}(T_{p,pn-1}) &= (n-1) \sum_{i=1}^k (p-2i) + \sum_{i=1}^k (p-2i-1) \\
&= (n-1) \sum_{i=1}^{k-1} 2i + \sum_{i=1}^{k-1} 2i-1 \\
&= (n-1)(k-1)k + (k-1)^2 \\
&= nk^2 - nk - k + 1 \\
&= n \left\lfloor \frac{(p-1)^2}{4} \right\rfloor - \left\lfloor \frac{p-1}{2} \right\rfloor.
\end{aligned}$$

This proves the theorem for  $T_{p,pn+1}$ . A similar analysis yields the knot Floer width for the knots  $T_{5,5n+2}$  and  $T_{5,5n+3}$ .  $\square$

#### 4. TURAEV GENUS MINIMIZING DIAGRAMS OF $T_{p,q}$

In this section, we construct diagrams of  $T_{p,q}$  whose Turaev genus and dealternating number are equal to or just slightly larger than the lower bounds given by Theorems 1.3 and 3.1. We also prove Theorems 1.1 and 1.2.

Many of the diagrams in this section are in closed braid form. Let  $B_p$  denote the braid group on  $p$  strands, and let  $\sigma_i$  denote the braid where strand  $i+1$  passes over strand  $i$  as in Figure 4. The braid group  $B_p$  is generated by  $\sigma_i$  for  $i = 1, \dots, p-1$ . The relations in  $B_p$  come in two formats:

$$(4.1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \text{ and}$$

$$(4.2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ if } i = 1, \dots, p-2.$$

Since the braid words are rather long, we adopt the following convention. The braid generator  $\sigma_i$  will be denoted by the integer  $i$ . A product of braid generators  $\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$  is represented by the string  $i_1 i_2 \cdots i_k$ . A power of braid generators  $(\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k})^j$  is represented by  $(i_1 i_2 \cdots i_k)^j$ . We only use positive braid generators in this article.

The torus knot  $T_{p,q}$  is the closure of the braid  $(123 \cdots p-1)^q$ . A full twist  $\Delta_p \in B_p$  is the braid  $(123 \cdots p-1)^p$ . The full twist  $\Delta_p$  is in the center of the braid group  $B_p$ . If  $q = pn + r$  where  $0 \leq r < p$ , then  $T_{p,q}$  is the closure of the braid  $\Delta_p^n (123 \cdots p-1)^r$ . Alternate forms of the full twists  $\Delta_4$ ,  $\Delta_5$ , and  $\Delta_6$  are the building blocks for the proofs of Theorems 1.1 and 1.2. Our strategy is

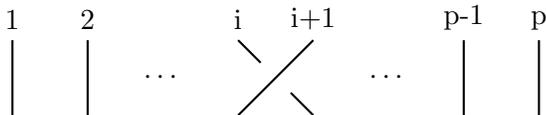


Figure 4: The braid generator  $\sigma_i$  in  $B_p$ , the braid group on  $p$  strands .

to first find a diagram of the closure of  $\Delta_p$  whose Turaev surface has small genus, then to do the same for the closure of  $(\Delta_p)^n$ , and finally for the closure of  $\Delta_p^n(123 \cdots p-1)^r$ .

For any link diagram  $D$ , the inequality  $g_T(D) \leq \text{dalt}(D)$  holds. So a strategy that minimizes  $\text{dalt}(D)$  will give diagrams of small Turaev genus as well. Since all of our braids are positive, changing the even indexed crossings ( $\sigma_2$  in  $T_{4,q}$  and  $\sigma_2$  and  $\sigma_4$  in  $T_{5,q}$  and  $T_{6,q}$ ) will result in an alternating diagram. Work from [Low11] implies that replacing any crossing with a positive power of that crossing does not change the genus of the Turaev surface. This suggests our strategy to minimize the Turaev genus of the closure of  $\Delta_p$  should be to find a diagram with the fewest number of groups of even indexed crossings.

Figures 5, 6, and 7 show that

$$(4.3) \quad \Delta_4 = 113321322132,$$

$$(4.4) \quad \Delta_5 = (2311234311)^2, \text{ and}$$

$$(4.5) \quad \Delta_6 = 2\zeta 234\eta 43,$$

where

$$(4.6) \quad \zeta = 133545332334513 \in B_6 \text{ and}$$

$$(4.7) \quad \eta = 315531213 \in B_6.$$

In Figures 5, 6, and 7, the shaded regions indicate the portions of the diagram that will be changed to obtain the subsequent diagram.

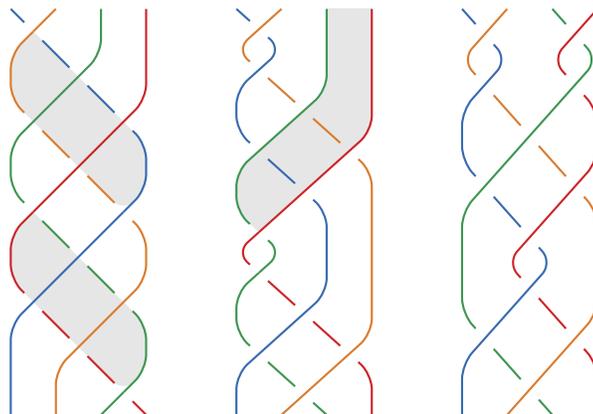
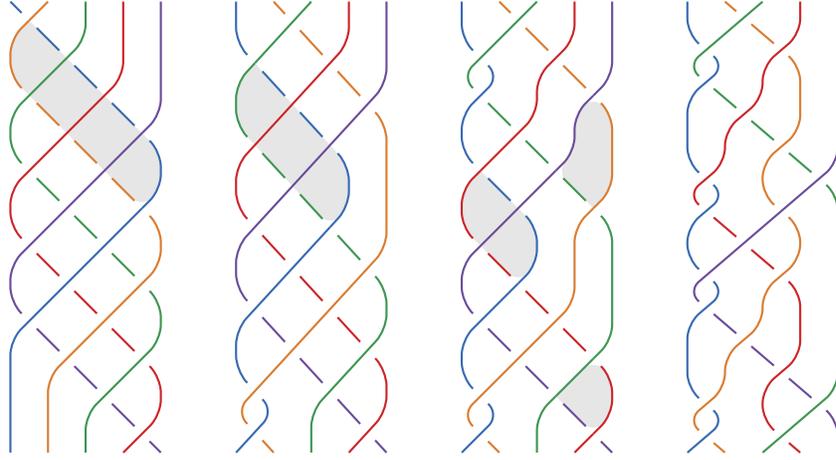
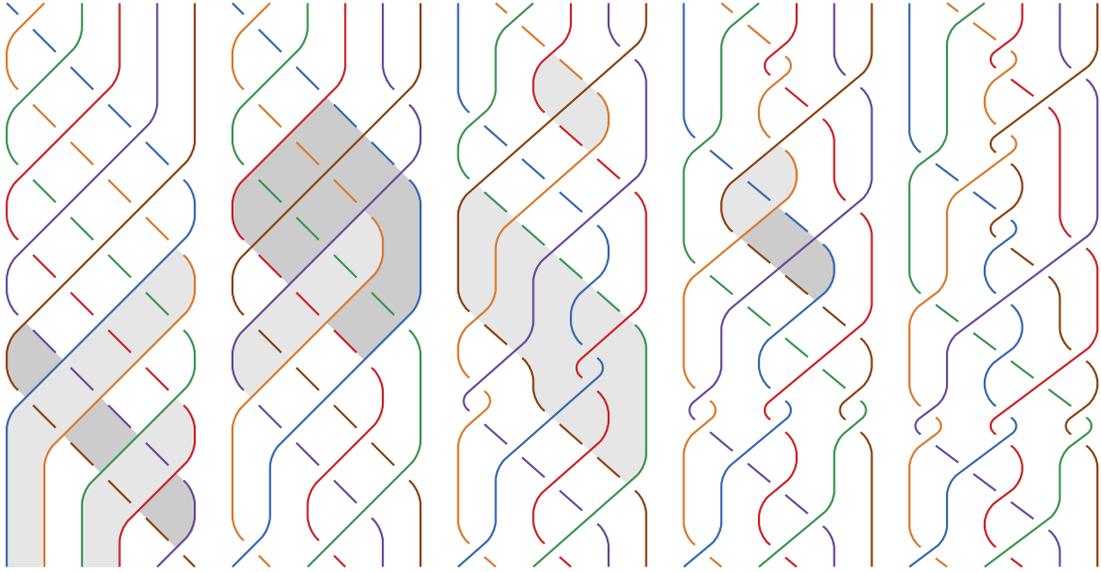


Figure 5: Transforming a full twist  $\Delta_4$  on 4 strands into 113321322132.

Lemmas 4.1, 4.2, and 4.3 give equivalent braid words for all torus links on four or five strands and the torus links of the form  $T_{6,6n}$  and  $T_{6,6n+1}$ . The main computational tools are Equation 4.1

Figure 6: Transforming a full twist  $\Delta_5$  on 5 strands into  $(2311234311)^2$ .Figure 7: Transforming a full twist  $\Delta_6$  on 6 strands into  $2\zeta 234\eta 43$ .

and 4.2. However, in the case for  $T_{5,5n+3}$ , we need one additional tool, namely cyclic permutation of the braid word. While this can change the element of the braid group, it does not change the link type of the closure. If two braids  $\beta_1$  and  $\beta_2$  are related by a cyclic permutation of the braid word, we write  $\beta_1 \equiv \beta_2$ .

The computations in these proofs can involve lengthy braid words. In order to guide the reader, we adopt the following conventions. If a word  $w$  is to be replaced by an equivalent word using the braid relation, we underline the word  $\underline{w}$ . If we swap two commuting words  $w_1$  and  $w_2$ , then we

indicate the move by  $\overrightarrow{w_1} \overleftarrow{w_2}$ . If a word  $w$  is to be replaced by an equivalent word coming from a previous computation or the inductive hypothesis, then we indicate it by  $\underline{w}$ .

**Lemma 4.1.** *For each positive integer  $n$ , the following equalities hold in the braid group  $B_4$ :*

- $(123)^{4n} = 1^{2n} 3^{2n} (2132)^{2n}$ ,
- $(123)^{4n+1} = 1^{2n} 3^{2n} (2132)^{2n-1} 2131213$ ,
- $(123)^{4n+2} = 1^{2n+2} 3^{2n} (2132)^{2n+1}$ , and
- $(123)^{4n+3} = 1^{2n+2} 3^{2n} (2132)^{2n} 2131213$ .

*Proof.* We prove that  $(123)^{4n} = 1^{2n} 3^{2n} (2132)^{2n}$  by induction on  $n$ . In the case where  $n = 1$ , Figure 5 shows that  $(123)^4 = 113321322132$ . Recall that  $(123)^4$  is in the center of  $B_4$ . Hence

$$\begin{aligned}
(123)^{4(n+1)} &= \underbrace{(123)^{4n}}_{\sim} (123)^4 \\
&= 1^{2n} 3^{2n} \underbrace{(2132)^{2n}}_{\sim} \underbrace{(123)^4}_{\sim} \\
&= 1^{2n} 3^{2n} \underbrace{(123)^4}_{\sim} (2132)^{2n} \\
&= 1^{2n} \underbrace{3^{2n}}_{\sim} \underbrace{1133213221132}_{\sim} (2132)^{2n} \\
&= 1^{2(n+1)} 3^{2(n+1)} (2132)^{2(n+1)}.
\end{aligned}$$

Figure 8 shows that  $11332132 = 21321133$ . Now since  $(123)^{4n}$  is in the center of  $B_4$ , we have

$$\begin{aligned}
(123)^{4n+1} &= \underbrace{(123)^{4n}}_{\sim} 123 \\
&= 1^{2n} 3^{2n} (2132)^{2n-1} 2132123 \\
&= 1^{2n} 3^{2n} (2132)^{2n-1} 2131213, \\
(123)^{4n+2} &= 123123 \underbrace{(123)^{4n}}_{\sim} \\
&= 12 \underbrace{3}_{\sim} \underbrace{1}_{\sim} 231^{2n} 3^{2n} (2132)^{2n} \\
&= 1213231^{2n} 3^{2n} (2132)^{2n} \\
&= 1212321^{2n} 3^{2n} (2132)^{2n} \\
&= 1121321^{2n} 3^{2n} (2132)^{2n} \\
&= 1^{2n+2} 3^{2n} (2132)^{2n+1}, \text{ and} \\
(123)^{4n+3} &= \underbrace{(123)^{4n+2}}_{\sim} 123 \\
&= 1^{2n+2} 3^{2n} (2132)^{2n} 2132123 \\
&= 1^{2n+2} 3^{2n} (2132)^{2n} 2131213.
\end{aligned}$$

□

**Lemma 4.2.** *Let  $\alpha, \beta, \gamma \in B_5$  be defined by*

$$\alpha = 323311, \quad \beta = 31123112334311, \quad \text{and} \quad \gamma = 31123112311.$$

*For each positive integer  $n$ , the following equalities hold in  $B_5$ :*

- $(1234)^{5n} = 2311\alpha^{n-1}234\beta^{n-1}\gamma43$ ,

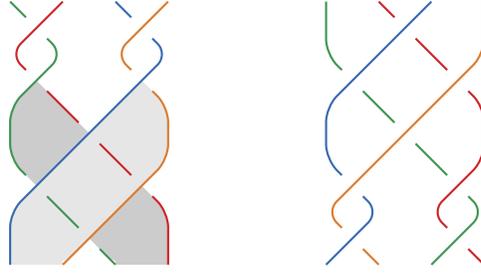


Figure 8: The braid words 1133 and 2132 commute. Thus  $11332132 = 21321133$ .

- $(1234)^{5n+1} = 1323311\alpha^{n-1}234\beta^{n-1}\gamma343$ ,
- $(1234)^{5n+2} = 1231323311\alpha^{n-1}234\beta^{n-1}\gamma3433$ ,
- $(1234)^{5n+3} \equiv 12131231323311\alpha^{n-1}234\beta^{n-1}\gamma3433$ , and
- $(1234)^{5n+4} = 2311\alpha^n234\beta^n311231422$ .

*Proof.* We show that  $(1234)^{5n} = 2311\alpha^{n-1}234\beta^{n-1}\gamma43$  by induction on  $n$ . For  $n = 1$ , Figure 6 shows that  $(1234)^5 = (2311234311)^2$ . We have

$$\begin{aligned}
 (1234)^{5(n+1)} &= \underbrace{(1234)^{5n}}_{\leftarrow} \underbrace{(1234)^5}_{\rightarrow} \\
 &= 2311\alpha^{n-1}234\beta^{n-1}\gamma43 \underbrace{(2311234311)^2}_{\leftarrow} \\
 &= 2311\alpha^{n-1} \underline{2323112343112311234311} \underline{4} \beta^{n-1}\gamma43 \\
 &= 2311\alpha^{n-1} \underline{32331123431123112343411} \beta^{n-1}\gamma43 \\
 &= 2311\alpha^{n-1} \underline{32331123431123112334311} \beta^{n-1}\gamma43 \\
 &= 2311\alpha^{n-1} \alpha 234\beta\beta^{n-1}\gamma43 \\
 &= 2311\alpha^n234\beta^n\gamma43.
 \end{aligned}$$

Also,

$$\begin{aligned}
 (1234)^{5n+1} &= \underbrace{(1234)^{5n}}_{\leftarrow} \underbrace{(1234)}_{\rightarrow} \\
 &= 123 \underbrace{(1234)^{5n}}_{\leftarrow} 4 \\
 &= 1232311\alpha^{n-1}234\beta^{n-1}\gamma \underline{434} \\
 &= 1323311\alpha^{n-1}234\beta^{n-1}\gamma343.
 \end{aligned}$$

Figure 9 shows that  $12341234 = 12312343$ . Thus

$$\begin{aligned}
 (1234)^{5n+2} &= (1234)^{5n} \underbrace{(12341234)}_{\leftarrow} \\
 &= \underbrace{(1234)^{5n}}_{\leftarrow} \underbrace{(12312343)}_{\rightarrow} \\
 &= 123123 \underbrace{(1234)^{5n}}_{\leftarrow} 43 \\
 &= 1231 \underline{232311} \alpha^{n-1} 234 \beta^{n-1} \gamma \underline{4343} \\
 &= 1231323311\alpha^{n-1}234\beta^{n-1}\gamma3433.
 \end{aligned}$$

Figure 10 shows that  $(1234)^3 = (123)^3 432$ . Thus

$$\begin{aligned}
(1234)^{5n+3} &= (1234)^{5n} \underbrace{(1234)^3}_{\leftarrow \rightarrow} \\
&= \underbrace{(1234)^{5n}}_{\leftarrow \rightarrow} (123)^3 432 \\
&= (123)^3 \underbrace{(1234)^{5n}}_{\leftarrow \rightarrow} 432 \\
&= 1231231 \underline{232311} \alpha^{n-1} 234 \beta^{n-1} \gamma \underline{43432} \\
&= 1231231323311 \alpha^{n-1} 234 \beta^{n-1} \gamma 3433 \underline{2} \\
&\equiv \underline{21231231323311} \alpha^{n-1} 234 \beta^{n-1} \gamma 3433 \\
&= 12131231323311 \alpha^{n-1} 234 \beta^{n-1} \gamma 3433.
\end{aligned}$$

Also,

$$\begin{aligned}
\underbrace{(1234)^{5(n+1)}}_{\leftarrow \rightarrow} &= 2311 \alpha^n 234 \beta^n 31123112 \underline{3} \underline{1143} \\
&= 2311 \alpha^n 234 \beta^n 311231121134 \underline{3} \\
&= 2311 \alpha^n 234 \beta^n 3112311211 \underline{4} 34 \\
&= 2311 \alpha^n 234 \beta^n 3112314121134 \\
&= 2311 \alpha^n 234 \beta^n 3112314212134 \\
&= 2311 \alpha^n 234 \beta^n 3112314221234.
\end{aligned}$$

Therefore

$$(1234)^{5n+4} = 2311 \alpha^n 234 \beta^n 311231422.$$

□

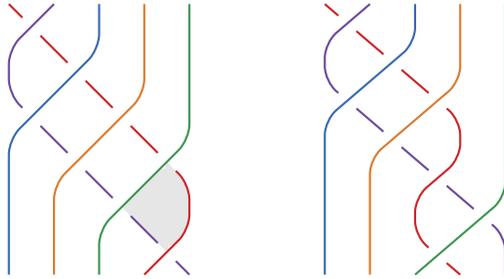


Figure 9: In  $B_5$  the equality  $12341234 = 12312343$  holds.

**Lemma 4.3.** Let  $\zeta, \eta \in B_6$  be defined by

$$\zeta = 133545332334513 \text{ and } \eta = 315531213.$$

For each positive integer  $n$ , the following equalities hold in the braid group  $B_6$ :

- $(12345)^{6n} = 2(\zeta 323)^{n-1} \zeta 234 (\eta 343)^{n-1} \eta 43$ ,
- $(12345)^{6n+1} = 1323 (\zeta 323)^{n-1} \zeta 234 (\eta 343)^{n-1} \eta 3435$ ,

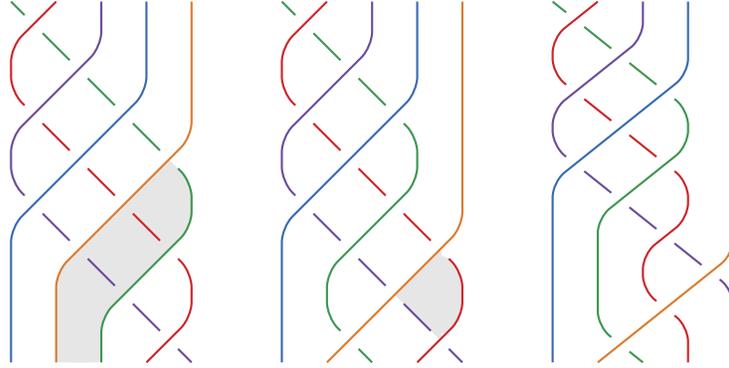


Figure 10: In  $B_5$  the equality  $(1234)^3 = (123)^3 432$  holds.

*Proof.* We show the first equality by induction on  $n$ . If  $n = 1$ , then Figure 7 shows that  $(12345)^6 = 2\zeta 234\eta 43$ . We have

$$\begin{aligned}
 (12345)^{6(n+1)} &= \underbrace{(12345)^{6n}}_{\leftarrow} \underbrace{(12345)^6}_{\rightarrow} \\
 &= 2(\zeta 323)^{n-1} \zeta 234 (\eta 343)^{n-1} \eta 432 \zeta 234 \eta 43 \\
 &= 2(\zeta 323)^{n-1} \zeta 232 \zeta 234 \eta 434 (\eta 343)^{n-1} \eta 43 \\
 &= 2(\zeta 323)^{n-1} \zeta 323 \zeta 234 \eta 343 (\eta 343)^{n-1} \eta 43 \\
 &= 2(\zeta 323)^n \zeta 234 (\eta 343)^n \eta 43,
 \end{aligned}$$

as desired. Also,

$$\begin{aligned}
 (12345)^{6n+1} &= \underbrace{(12345)^{6n}}_{\leftarrow} \underbrace{(12345)}_{\rightarrow} \\
 &= 123 \underbrace{(12345)^{6n}}_{\rightarrow} 45 \\
 &= 1232(\zeta 323)^{n-1} \zeta 234 (\eta 343)^{n-1} \eta 4345 \\
 &= 1323(\zeta 323)^{n-1} \zeta 234 (\eta 343)^{n-1} \eta 3435.
 \end{aligned}$$

□

Let  $D_{p,q}$  be the diagram of  $T_{p,q}$  appearing on the right hand side in Lemmas 4.1, 4.2, and 4.3. Propositions 4.4, 4.5, and 4.6 compute the genus of the Turaev surface for each such  $D_{p,q}$ .

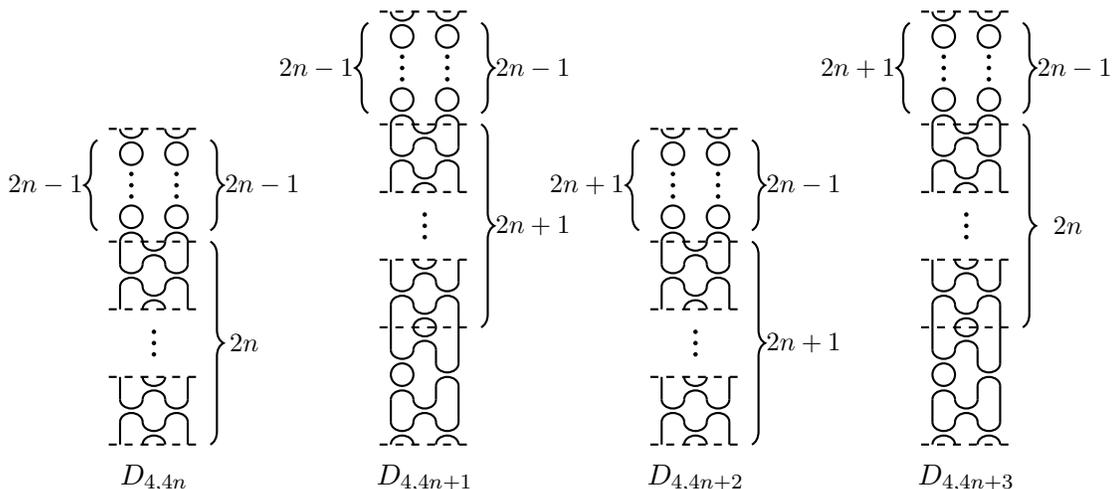
**Proposition 4.4.** *Let  $D_{4,4n+i}$  be the closure of the braid diagrams appearing on the right hand side of Lemma 4.1 for  $i = 0, 1, 2$ , and 3. Then*

$$g_T(D_{4,4n}) = g_T(D_{4,4n+1}) = 2n \text{ and } g_T(D_{4,4n+2}) = g_T(D_{4,4n+3}) = 2n + 1.$$

*Proof.* The diagram  $D_{4,4n+i}$  has  $12n + 3i$  crossings. The number of components in the all- $A$  state of  $D_{4,4n+i}$  is  $s_A(D_{4,4n+i}) = 4$ . Figure 11 shows the all- $B$  states of  $D_{4,4n+i}$ . From that figure, one can conclude

$$s_B(D_{4,4n}) = 8n - 2, \quad s_B(D_{4,4n+1}) = 8n + 1, \quad s_B(D_{4,4n+2}) = 8n + 2, \quad \text{and } s_B(D_{4,4n+3}) = 8n + 5.$$

Equation 2.1 then implies the result.

Figure 11: The all- $B$  states of  $D_{4,4n+i}$  for  $i = 0, 1, 2,$  and  $3$ .

□

**Proposition 4.5.** *Let  $D_{5,5n+i}$  be the closure of the braid diagrams appearing on the right hand side of Lemma 4.2 for  $i = 0, 1, 2, 3,$  and  $4$ . Then*

$$g_T(D_{5,5n}) = g_T(D_{5,5n+1}) = 4n \text{ and } g_T(D_{5,5n+j}) = 4n + j - 1$$

for  $j = 2, 3,$  and  $4$ .

*Proof.* The diagram  $D_{5,5n+i}$  has  $20n + 4i$  crossings. The number of components in the all- $A$  state of  $D_{5,5n+i}$  is  $s_A(D_{5,5n+i}) = 5$ . Since the expressions for  $D_{5,5n+i}$  in Lemma 4.2 contain  $\alpha^{n-1}$  and  $\beta^{n-1}$  for  $i = 0, 1, 2,$  and  $3$ , we must handle the case where  $n = 1$  separately from the case where  $n > 1$ . Figure 12 shows the all- $B$  states of  $D_{5,5}, D_{5,6}, D_{5,7},$  and  $D_{5,8}$ . From that figure one can see that

$$s_B(D_{5,5}) = 9, \quad s_B(D_{5,6}) = 13, \quad s_B(D_{5,7}) = 15, \quad \text{and } s_B(D_{5,8}) = 17.$$

Equation 2.1 then implies that

$$g_T(D_{5,5}) = g_T(D_{5,4}) = 4, \quad g_T(D_{5,7}) = 5, \quad \text{and } g_T(D_{5,8}) = 6.$$

If  $q \geq 9$ , then  $D_{5,q}$  contains at least one  $\alpha$  and  $\beta$  as a sub-word in its braid word. Figure 13 shows that consecutive  $\alpha$  words add 4 components to the all- $B$  state and consecutive  $\beta$  words add 8 components to the all- $B$  state. This observation allows us to compute the number of components in the all- $B$  state of  $D_{5,q}$  by only considering braid words with  $\alpha$  and  $\beta$  appearing once. A braid word where  $\alpha^\ell$  and  $\beta^m$  are replaced by  $\alpha$  and  $\beta$  respectively will be called *reduced*.

Figure 14 shows the reduced all- $B$  states of  $D_{5,q}$  for  $q \geq 9$ . If  $n \geq 2$ , then the reduced all- $B$  state of  $D_{5,5n}$  has 21 components. Hence  $s_B(D_{5,5n}) = 21 + 12(n-2) = 12n - 3$ . Equation 2.1 implies that  $g_T(D_{5,5n}) = 4n$ . If  $n \geq 2$ , then the reduced all- $B$  state of  $D_{5,5n+1}$  has 25 components. Hence  $s_B(D_{5,5n+1}) = 25 + 12(n-2) = 12n + 1$ . Equation 2.1 implies that  $g_T(D_{5,5n+1}) = 4n$ . If  $n \geq 2$ , then the reduced all- $B$  state of  $D_{5,5n+2}$  has 27 components. Hence  $s_B(D_{5,5n+2}) = 27 + 12(n-2) = 12n + 3$ . Equation 2.1 implies that  $g_T(D_{5,5n+2}) = 4n + 1$ . If  $n \geq 2$ , then the reduced all- $B$  state of  $D_{5,5n+3}$  has 29 components. Hence  $s_B(D_{5,5n+3}) = 29 + 12(n-2) = 12n + 5$ . Equation 2.1 implies that

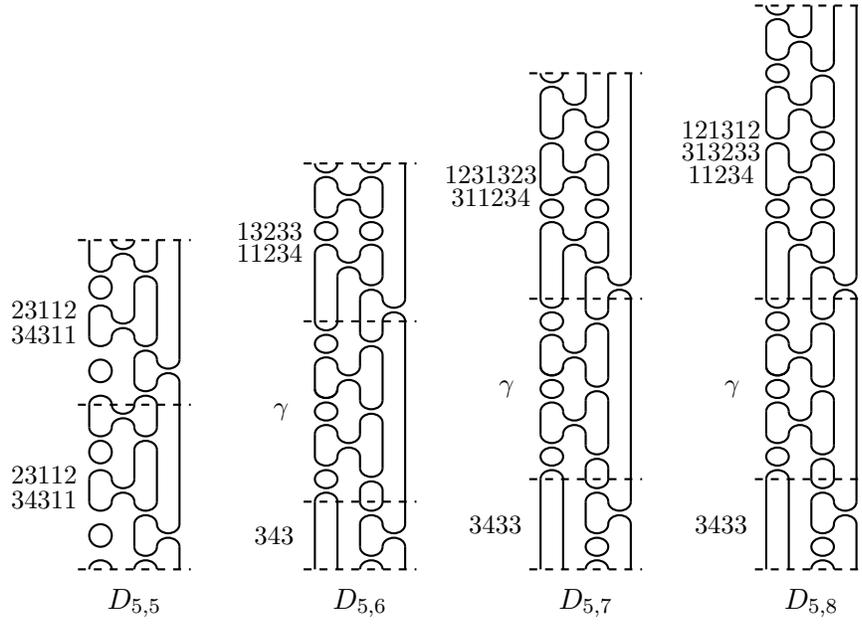


Figure 12: The all- $B$  states of  $D_{5,5+i}$  for  $i = 0, 1, 2,$  and  $3$ .

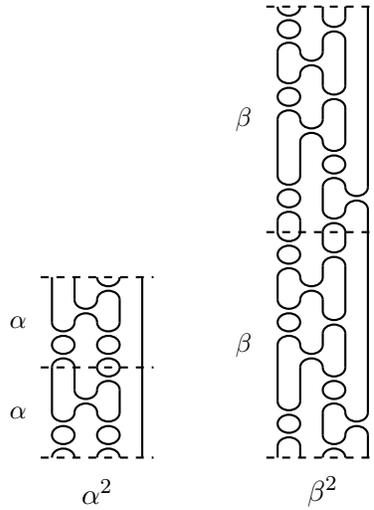


Figure 13: Consecutive  $\alpha$  braids adds 3 components to the all- $B$  state. Consecutive  $\beta$  braids adds 8 components to the all- $B$  state.

$g_T(D_{5,5n+3}) = 4n + 2$ . Finally, if  $n \geq 1$ , then the reduced all- $B$  state of  $D_{5,5n+4}$  has 19 components. Hence  $s_B(D_{5,5n+4}) = 19 + 12(n - 1) = 12n + 7$ . Equation 2.1 implies that  $g_T(D_{5,5n+4}) = 4n + 3$ .  $\square$

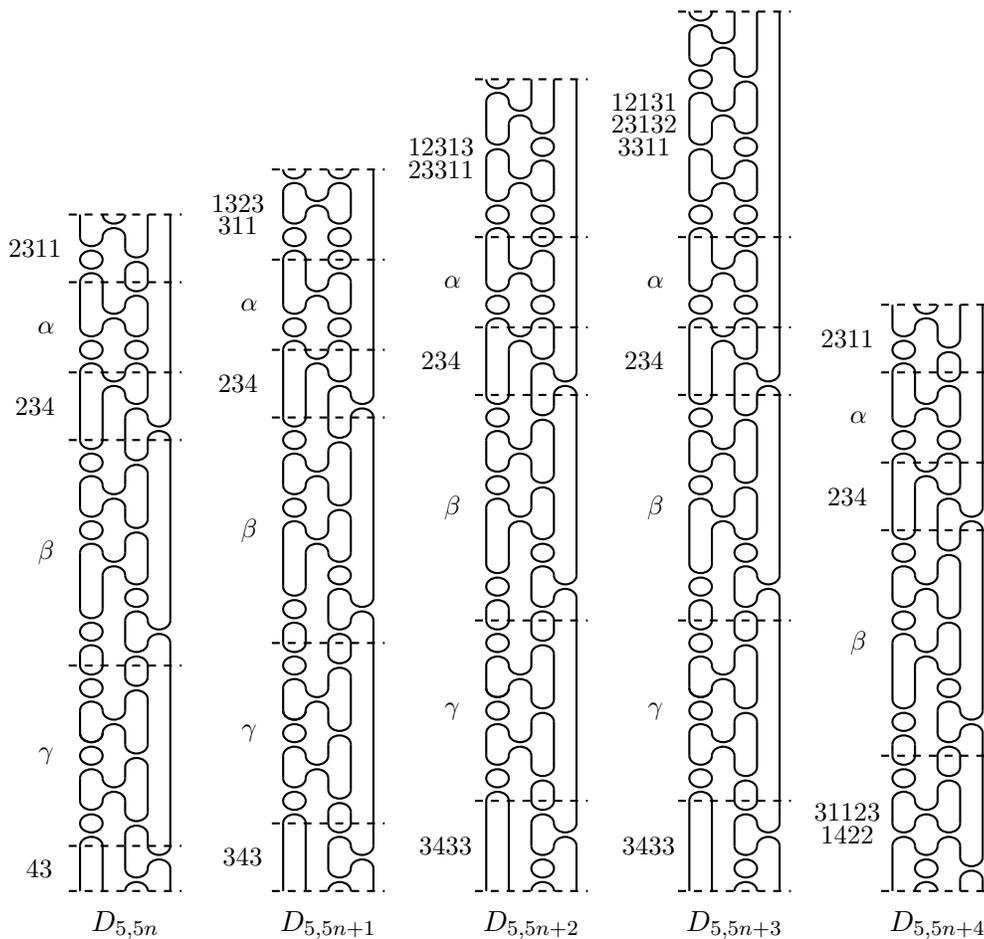


Figure 14: The reduced all- $B$  states of  $D_{5,5n+i}$  for  $i = 0, 1, 2, 3$ , and  $4$ .

**Proposition 4.6.** *Let  $D_{6,6n}$  and  $D_{6,6n+1}$  be the closure of the braid diagrams appearing on the right hand side of Lemma 4.3. Then*

$$g_T(D_{6,6n}) = g_T(D_{6,6n+1}) = 6n.$$

*Proof.* For  $i = 0$  and  $1$ , the diagram  $D_{6,6n+i}$  has  $30n + 5i$  crossings. The number of components in the all- $A$  state of  $D_{6,6n+i}$  is  $s_A(D_{6,6n+i}) = 6$ . Since the expressions for  $D_{6,6n+i}$  in Lemma 4.3 contain  $(\alpha 323)^{n-1}$  and  $(\beta 343)^{n-1}$ , we must handle the case where  $n = 1$  separately from the case where  $n > 1$ . Figure 15 shows the all- $B$  states of  $D_{6,6}$  and  $D_{6,7}$ . From that figure one can see that

$$s_B(D_{6,6}) = 14, \text{ and } s_B(D_{6,7}) = 19$$

Equation 2.1 then implies that

$$g_T(D_{6,6}) = g_T(D_{6,7}) = 6.$$

If  $n \geq 2$ , then  $D_{6,6n}$  and  $D_{6,6n+1}$  contains at least one  $\alpha 323$  and  $\beta 343$  as a sub-word in its braid word. A straightforward computation shows that consecutive  $\alpha 323$  words add 10 components to the all- $B$  state and consecutive  $\beta 343$  words add 8 components to the all- $B$  state. This observation

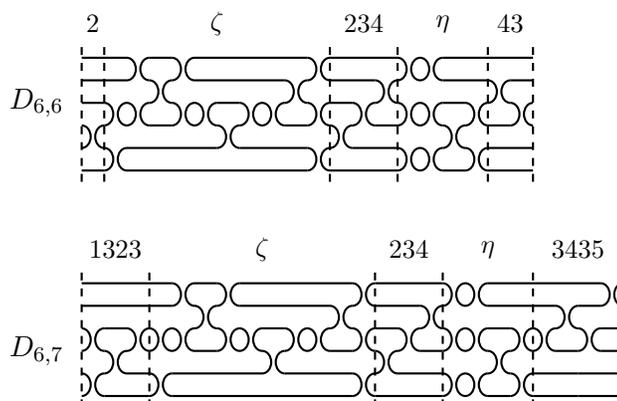


Figure 15: The all- $B$  states of  $D_{6,6}$  and  $D_{6,7}$ .

allows us to compute the number of components in the all- $B$  state of  $D_{6,6n}$  and  $D_{6,6n+1}$  by only considering braid words with  $\alpha 323$  and  $\beta 343$  appearing once. A braid word where  $(\alpha 323)^\ell$  and  $(\beta 343)^m$  are replaced by  $\alpha 323$  and  $\beta 343$  respectively will be called *reduced*.

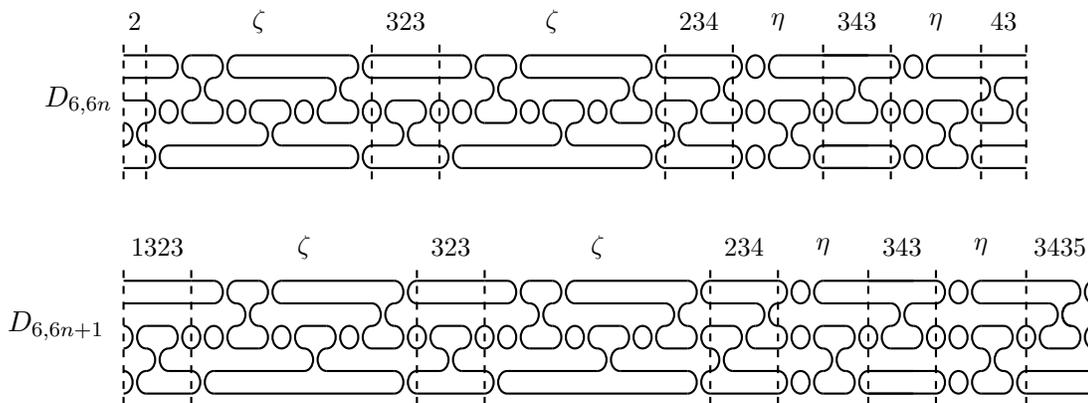


Figure 16: The reduced all- $B$  states of  $D_{6,6n}$  and  $D_{6,6n+1}$  for  $n \geq 2$ .

Figure 16 shows the reduced all- $B$  states of  $D_{6,6n}$  and  $D_{6,6n+1}$  for  $n \geq 2$ . The reduced all- $B$  state of  $D_{6,6n}$  has 32 components. Hence  $s_B(D_{6,6n}) = 32 + 18(n - 2) = 18n - 4$ . Equation 2.1 implies that  $g_T(D_{6,6n}) = 6n$ . The reduced all- $B$  state of  $D_{6,6n+1}$  has 37 components. Hence  $s_B(D_{6,6n+1}) = 37 + 18(n - 2) = 18n + 1$ . Equation 2.1 implies that  $g_T(D_{6,6n+1}) = 6n$ . □

*Proof of Theorem 1.1.* Suppose that  $n = 0$ . Since  $T_{4,1}$ ,  $T_{5,1}$ , and  $T_{6,1}$  are all unknots, it follows that their Turaev genera are all zero. Moreover, since  $T_{4,3} = T_{3,4}$ , we have  $g_T(T_{4,3}) = 1$ .

If  $n > 0$ , then the result is implied by Theorem 1.3 and Propositions 4.4, 4.5, and 4.6. □

Propositions 4.4, 4.5, and 4.6 also immediately imply the following theorem.

**Theorem 4.7.** For  $n \geq 1$  and  $j = 2, 3$ , and  $4$ , we have

$$\begin{aligned} g_T(T_{4,4n}) &\leq 2n, & g_T(T_{4,4n+2}) &\leq 2n + 1, \\ g_T(T_{5,5n}) &\leq 4n, & g_T(T_{6,6n}) &\leq 6n. \end{aligned}$$

The diagrams in Lemmas 4.1, 4.2, and 4.3 are the starting points for the proof of Theorem 1.2.

*Proof of Theorem 1.2.* In any of the braid words from Lemmas 4.1, 4.2, and 4.3, changing the even indexed crossings (i.e.  $\sigma_2$  in  $T_{4,q}$  and  $\sigma_2$  and  $\sigma_4$  in  $T_{5,q}$  and  $T_{6,q}$ ) gives an alternating diagram. This process yields  $\text{dalt}(D_{6,6n+1}) = 6n + 2$ . Thus  $6n \leq \text{dalt}(T_{6,6n+1}) \leq 6n + 2$ . Applying the above process to the 4 and 5 stranded torus knots does not lead to the smallest possible dealternating number; we will introduce two tricks to improve the result.

The first trick that we use is to replace a twist region of even indexed crossings with the same region encircled by one of its incoming strands (as in Figure 17). Let  $D'_{4,4n+1}$  and  $D'_{4,4n+3}$  be the diagrams  $D_{4,4n+1}$  and  $D_{4,4n+3}$  with each  $\sigma_2$ -twist region modified as above. Changing the crossings at the bottom right of each modified twist region and changing the isolated  $\sigma_2$  crossing results in an alternating diagram. Thus the dealternating number of  $D'_{4,4n+1}$  and  $D'_{4,4n+3}$  is the number of isolated  $\sigma_2$  crossings plus the number of  $\sigma_2$  twist regions, and hence  $\text{dalt}(D'_{4,4n+1}) = 2n + 1$  and  $\text{dalt}(D'_{4,4n+3}) = 2n + 3$ . Theorem 1.3 implies that  $2n \leq \text{dalt}(T_{4,4n+1}) \leq 2n + 1$  and  $2n + 1 \leq \text{dalt}(T_{4,4n+3}) \leq 2n + 2$ .

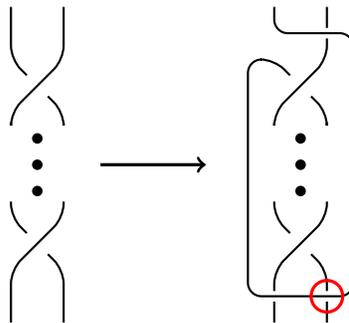


Figure 17: Changing the circled crossing on the lower right makes this region alternating.

The second trick we use is wrap a strand of the link between the  $\sigma_3$  and  $\sigma_4$  crossings in the diagram  $D_{5,5n+1}$  to obtain the diagram  $D'_{5,5n+1}$ , as in Figure 18. The diagram  $D_{5,5n+1}$  has  $n + 1$   $\sigma_4$ -crossings. The strand between the  $\sigma_3$  and  $\sigma_4$  crossings has  $2n + 1$  crossings. Changing  $n$  of those crossings results in an alternating diagram. Thus the dealternating number of  $D'_{5,5n+1}$  is  $n$  plus the number of  $\sigma_2$  crossings, yielding  $\text{dalt}(D'_{5,5n+1}) = 4n + 1$ . Theorem 1.3 implies that  $4n \leq \text{dalt}(T_{5,5n+1}) \leq 4n + 1$ . The same strategy yields the result for the remaining torus knots on 5 strands.  $\square$

The methods of the above proof can be used to prove the following result. The details are omitted.

**Theorem 4.8.** For  $n \geq 1$ , we have

$$\begin{aligned} \text{dalt}(T_{4,4n}) &\leq 2n + 1, & \text{dalt}(T_{4,4n+2}) &\leq 2n + 2, \\ \text{dalt}(T_{5,5n}) &\leq 4n + 1, & \text{dalt}(T_{6,6n}) &\leq 6n + 2. \end{aligned}$$

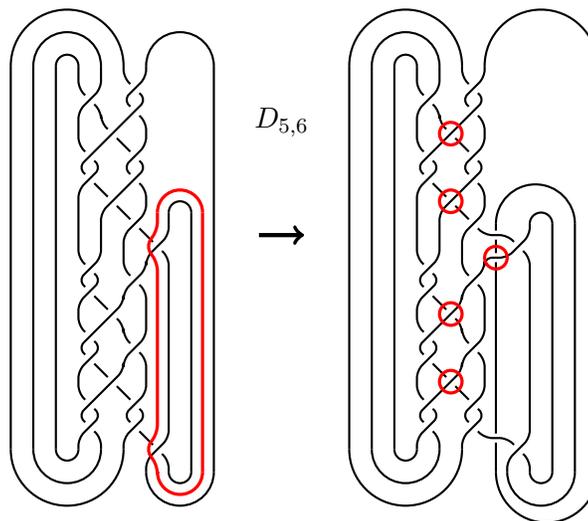


Figure 18: Wrapping the strand around the  $\sigma_4$  crossings decreases the dealternating number of the diagram by one.

As Theorem 1.1 implies, we were able to find Turaev genus minimizing diagrams for many but not all of the torus knots on six or fewer strands. In the cases where we did not compute the Turaev genus exactly, there are several possibilities. The Turaev genus of these knots could be strictly greater than the lower bound given by knot Floer homology. Alternatively, since our search of diagrams only considered the closures of positive braids, it is possible that the Turaev genus minimizing diagrams of our unsolved cases include negative generators  $\sigma_i^{-1}$  or possibly are not even closures of braids. It remains an interesting question to compute the Turaev genus or dealternating numbers of an arbitrary torus knot  $T_{p,q}$ .

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DEPARTMENT OF MATHEMATICS AND STATISTICS, VASSAR COLLEGE, POUGHKEEPSIE, NY  
*E-mail address:* [kajin@vassar.edu](mailto:kajin@vassar.edu)

DEPARTMENT OF MATHEMATICS AND STATISTICS, VASSAR COLLEGE, POUGHKEEPSIE, NY  
*E-mail address:* [adlowrance@vassar.edu](mailto:adlowrance@vassar.edu)

DEPARTMENT OF MATHEMATICS AND STATISTICS, VASSAR COLLEGE, POUGHKEEPSIE, NY  
*E-mail address:* [elpolston@vassar.edu](mailto:elpolston@vassar.edu)

DEPARTMENT OF MATHEMATICS AND STATISTICS, VASSAR COLLEGE, POUGHKEEPSIE, NY  
*E-mail address:* [yazheng@vassar.edu](mailto:yazheng@vassar.edu)